

# CIRCLE-VALUED MORSE THEORY FOR FRAME SPUN KNOTS AND SURFACE-LINKS

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ABSTRACT. Let  $N^k \subset S^{k+2}$  be a closed oriented submanifold, denote its complement by  $C(N) = S^{k+2} \setminus N$ . Denote by  $\xi \in H^1(C(N))$  the class dual to  $N$ . The Morse-Novikov number of  $C(N)$  is by definition the minimal possible number of critical points of a regular Morse map  $C(N) \rightarrow S^1$  belonging to  $\xi$ . In the first part of this paper we study the case when  $N$  is the twist frame spun knot associated to an  $m$ -knot  $K$ . We obtain a formula which relates the Morse-Novikov numbers of  $N$  and  $K$  and generalizes the classical results of D. Roseman and E.C. Zeeman about fibrations of spun knots. In the second part we apply the obtained results to the computation of Morse-Novikov numbers of surface-links in 4-sphere.

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## 1. INTRODUCTION

**1.1. Overview of the article.** Let  $N^k \subset S^{k+2}$  be a closed oriented submanifold, let  $C(N) = S^{k+2} \setminus N$  be its complement. The orientation of  $N$  determines a cohomology class  $\xi \in H^1(C(N)) \approx [C(N), S^1]$ . We say that  $N$  is fibred if there is a Morse map  $f : C(N) \rightarrow S^1$  homotopic to  $\xi$  which is regular nearby  $N$  (see Def 1.1) and has no critical points. In general a Morse map  $C(N) \rightarrow S^1$  has some critical points, the minimal number of these critical points will be called *the Morse-Novikov number of  $N$*  and denoted  $\mathcal{MN}(C(N))$ .

In the first part of this paper we study this invariant in relation with constructions of spinning. The classical Artin's spinning construction [2] associates to each knot  $K \subset S^3$  a 2-knot  $S(K) \subset S^4$ . A twisted version of this construction is due to E.C. Zeeman [14]. In [12] D. Roseman introduced a *frame spinning* construction, and G. Friedman [4] gave a twisted version of generalized Roseman's construction to include twisting.

The input data for twist frame spinning construction is:

(TFS1) A closed manifold  $M^k \subset S^{m+k}$  with trivial (and framed ) normal bundle.

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(TFS2) An  $m$ -knot  $K^m \subset S^{m+2}$ .

(TFS3) A smooth map  $\lambda : M \rightarrow S^1$ .

To these data one associates an  $n$ -knot  $\sigma(M, K, \lambda)$ , where  $n = k + m$  (see Section 2). We prove in Section 2 the following formula:

$$(1) \quad \mathcal{MN}(C(\sigma(M, K, \lambda))) \leq \mathcal{MN}(C(K)) \cdot \mathcal{MN}(M, [\lambda])$$

(where  $\mathcal{MN}(M, [\lambda])$  is the minimal number of critical points of a map  $M \rightarrow S^1$  homotopic to  $\lambda$ .) If  $\lambda$  is null-homotopic, we have

$$\mathcal{MN}(C(\sigma(M, K))) \leq \mathcal{MN}(C(K)) \cdot \mathcal{M}(M)$$

(where  $\mathcal{M}(M)$  is the Morse number of  $M$ ). In particular, if  $K$  is fibred, then the framed spun knot  $\sigma(M, K)$  is fibred (A theorem due to D. Roseman [12]). If in formula (1) the map  $l : M \rightarrow S^1$  has no critical points, then the knot  $\sigma(M, K, \lambda)$  is fibred, and we recover the classical result of E.C. Zeeman [14]: for any knot its twist-spun knot is fibred. In section 3 we discuss a geometric construction which is related to spinning, namely rotation of a knot  $K^m \subset S^{m+2}$  around equatorial sphere  $\Sigma$  of  $S^{m+2}$ . The resulting submanifold  $R(K)$  is diffeomorphic to  $S^1 \times S^m$  and is sometimes called *spun torus* of  $K$ . We prove that

$$\mathcal{MN}(R(K)) \leq 2\mathcal{MN}(K) + 2.$$

Section 4 is about Morse-Novikov theory for surface-links. In Subsection 4.1 we introduce a related invariant of surface-links, namely the *saddle number*  $sd(F)$  (Definition 4.1) and prove the formula

$$(2) \quad \mathcal{MN}(C(F)) \leq 2sd(F) + \chi(F) - 2.$$

In Subsection 4.2 we discuss the case of spun knots. In subsection 4.3 we apply the results of Sections 2 and 3 and the formula (2) to determine the Morse-Novikov numbers of certain surface-links. In [13] K. Yoshikawa introduced a numerical invariant  $ch(F)$  of surface-links  $F$  and developed a method that allowed him to enumerate all the (weakly prime) surface-links  $F$  with  $ch(F) \leq 10$ . In Subsection 4.3 we compute the Morse-Novikov numbers of the majority of the oriented surface-links of the Yoshikawa's table.

**1.2. Basic definitions and lower bounds for Morse-Novikov numbers.** We start with the definition of a regular Morse map.

**Definition 1.1.** Let  $N^k \subset S^{k+2}$  be a closed oriented submanifold. Denote by  $\xi \in H^1(C(N)) \approx [C(N), S^1]$  the cohomology class dual to the orientation of  $N$ . A Morse map  $f : C(N) \rightarrow S^1$  is said to be *regular* if there is an orientation preserving  $C^\infty$  trivialisation

$$(3) \quad \Phi : T(N) \rightarrow N \times B^2(0, \epsilon)$$

of a tubular neighbourhood  $T(N)$  of  $N$  such that the restriction  $f|_{(T(N) \setminus N)}$  satisfies  $f \circ \Phi^{-1}(x, z) = z/|z|$ .

An  $f$ -gradient  $v$  of a regular Morse map  $f : C(N) \rightarrow S^1$  will be called *regular* if there is a  $C^\infty$  trivialisation (3) such that  $\Phi^*(v)$  equals  $(0, v_0)$  where  $v_0$  is the Riemannian gradient of the function  $z \mapsto z/|z|$ .

If  $f$  is a Morse map of a manifold to  $\mathbb{R}$  or to  $S^1$ , then we denote by  $m_p(f)$  the number of critical points of  $f$  of index  $p$ . The number of all critical points of  $f$  is denoted by  $m(f)$ .

**Definition 1.2.** The minimal number  $m(f)$  where  $f : C(N) \rightarrow S^1$  is a regular Morse map is called *the Morse-Novikov number of  $N$*  and denoted by  $\mathcal{MN}(C(N))$ .

To obtain lower bounds for numbers  $m_p(f)$  one uses the *Novikov homology*. Let  $L = \mathbb{Z}[t, t^{-1}]$ ; denote by  $\hat{L} = \mathbb{Z}((t))$  and  $\hat{L}_{\mathbb{Q}} = \mathbb{Q}((t))$  the rings of all series in one variable  $t$  with integer (respectively rational) coefficients and finite negative part. Recall that  $\hat{L}$  is a PID, and  $\hat{L}_{\mathbb{Q}}$  is a field. Consider the infinite cyclic covering  $\overline{C(N)} \rightarrow C(N)$ ; the Novikov homology of  $C(N)$  is defined as follows:

$$\widehat{H}_*(C(N)) = H_*(\overline{C(N)}) \otimes_{\hat{L}} \hat{L}.$$

The rank and torsion number of the  $\hat{L}$ -module  $\widehat{H}_*(C(N))$  will be denoted by  $\hat{b}_k(C(N))$ , respectively  $\hat{q}_k(C(N))$ . For any regular Morse function  $f$  there is a Novikov complex  $\mathcal{N}_*(f, v)$  over  $\hat{L}$  generated in degree  $k$  by critical points of  $f$  of index  $k$  and such that  $H_*(\mathcal{N}_*(f, v)) \approx \widehat{H}_*(C(N))$ . (see [11]). Therefore we have the Novikov inequalities

$$\sum_k \left( \hat{b}_k(C(N)) + \hat{q}_k(C(N)) + \hat{q}_{k-1}(C(N)) \right) \leq \mathcal{MN}(C(N)).$$

These inequalities, which are far from being exact in general, are however very useful in particular in the case of surface-links (see Section 4).

## 2. TWIST FRAME SPUN KNOTS

We start with a recollection of the twist frame spinning construction following [12], [5], [4]. See the input data (TFS1) – (TFS3) for this construction on the page 1. Let  $a \in K$ . Removing a small open disk  $D(a)$  from  $S^{m+2}$  we obtain an embedded (knotted) disk  $K_0$  in the disk  $D^{m+2} \approx S^{m+2} \setminus D(a)$ . We identify  $D^{m+2}$  with the standard Euclidean disk of radius 1 and center 0 in  $\mathbb{R}^{m+2}$ . We have the usual diffeomorphism

$$\chi : S^{m+1} \times [0, 1[ \xrightarrow{\approx} D^{m+2} \setminus \{0\}, \quad (x, t) \mapsto tx.$$

We can assume that  $K_0 \cap \partial D^{m+2}$  is the standard sphere  $S^{m-1}$  in  $\partial D^{m+2} = S^{m+1}$ . Moreover, we can assume that the intersection of  $K_0$  with a neighbourhood of  $\partial D^{m+2}$  is also standard, that is,

$$K_0 \cap \chi(S^{m+1} \times [1 - \epsilon, 1]) = \chi(S^{m-1} \times [1 - \epsilon, 1]).$$

We have a framing of  $M$  in  $S^n$ ; combining this with the standard framing of  $S^n$  in  $S^{n+2}$  we obtain a diffeomorphism

$$\Phi : N(M, S^{n+2}) \xrightarrow{\approx} M \times D^m \times D^2$$

where  $N(M, S^{n+2})$  is a regular neighbourhood of  $M$  in  $S^{n+2}$ . We can assume that the restriction of  $\Phi$  to  $N(M, S^n)$  gives a diffeomorphism

$$\Phi : N(M, S^n) \xrightarrow{\approx} M \times D^m \times \{0\},$$

induced by the given framing of  $M$ . The Euclidean disc  $D^{m+2}$  is a subset of  $D^m \times D^2$ , so that  $K_0 \subset D^m \times D^2$ .

For  $\theta \in S^1$  denote by  $R_\theta$  the rotation of  $D^2$  around its center. The disc  $D^{m+2} \subset D^m \times D^2$  is invariant with respect to this rotation as well as the intersection

of  $K_0$  with a small neighbourhood of  $\partial D^{m+2}$ . We have  $\Phi(S^n \cap N(M, S^{n+2})) = M \times D^m \times \{0\}$ . Let

$$Z = \{(x, y, z) \mid (y, z) \in R_{\lambda(x)}(K_0)\}.$$

This is an  $m$ -dimensional submanifold of  $M \times D^m \times D^2$ . We define  $\sigma(M, K, \lambda)$  as follows

$$\sigma(M, K, \lambda) = (S^{n+2} \setminus N(M, S^{n+2})) \cup \Phi^{-1}(Z).$$

This is the image of an embedded  $n$ -sphere, knotted in general.

### Examples and particular cases.

- 1) Let  $\dim M = 0$ , so that  $M$  is a finite set; denote by  $p$  its cardinality. Then the  $n$ -knot  $\sigma(M, K, \lambda)$  is equivalent to the connected sum of  $p$  copies of  $K$ .
- 2) If  $M$  is the equatorial circle of the sphere  $S^2$ , which is in turn considered as an equatorial sphere of  $S^4$ , and  $\lambda(x) = 1$ , we obtain the classical Artin's construction. If  $\lambda : S^1 \rightarrow S^1$  is a map of degree  $d$ , we obtain the Zeeman's twist-spinning construction [14].
- 3) If  $\lambda(x) = 1$  for all  $x \in M$  we obtain the Roseman's construction of spinning around the manifold  $M$  [12]. In this case we will denote  $\sigma(M, K, \lambda)$  by  $\sigma(M, K)$ .

### Theorem 2.1.

$$\mathcal{MN}(\sigma(M, K, \lambda)) \leq \mathcal{MN}(K) \cdot \mathcal{MN}(M, [\lambda]).$$

(where  $[\lambda] \in H^1(M, \mathbb{Z}) \approx [M, S^1]$  is the homotopy class of  $\lambda$ ).

*Proof.* We will be using the terminology from the above construction of  $\sigma(M, K, \lambda)$ . We have the standard fibration

$$\psi_0 : S^{n+2} \setminus S^n \rightarrow S^1$$

obtained from the canonical framing of  $S^n$  in  $S^{n+2}$ . Observe that the map  $\alpha = \psi_0 \circ \Phi^{-1}$  is defined by the following formula

$$\alpha(x, y, z) = \frac{z}{|z|}.$$

Let  $f : S^{m+2} \setminus K \rightarrow S^1$  be a Morse map. The restriction of  $f$  to the subset  $D^{m+2} \setminus K_0$  will be denoted by the same letter  $f$ . We can assume that the function  $f$  equals  $\alpha$  in a neighbourhood of  $\partial D^{m+2} = S^{m+1}$ . In particular in a neighbourhood of  $\partial D^{m+2}$  we have

$$f(R_\theta(p)) = f(p) + \theta, \quad \text{for } p \in S^{m+1} \setminus K_0.$$

Define a function  $g$  on  $M \times D^{m+2} \setminus Z$  by the following formula:

$$(4) \quad g(x, \xi) = f(R_{-\lambda(x)}(\xi)) + \lambda(x),$$

(where  $x \in M$ ,  $\xi \in D^{m+2}$ ). Define a function  $\psi$  on the complement  $S^{n+2} \setminus \sigma$  by the the following formula:

- 1) If  $p \notin N(M, S^{n+2})$ , then  $\psi(p) = \psi_0(p)$ .
- 2) If  $p \in N(M, S^{n+2})$ , then  $\psi(p) = g(\Phi^{-1}(p))$ .

We will now prove that if  $\lambda$  is a Morse map (this can be achieved by a small perturbation of  $\lambda$ ), then  $\psi$  is also a Morse map, and the number  $m(\psi)$  of its critical points satisfy

$$m(\psi) = m(\lambda) \cdot m(f).$$

All the critical points of  $\psi$  are in  $N(M, S^{n+2})$ . In this domain the function  $\psi$  is diffeomorphic to  $g$ , and the count of critical points of  $g$  is easily achieved with the help of the next lemma.

**Lemma 2.2.** *Let  $g_1 : N_1 \rightarrow S^1$ ,  $g_2 : N_2 \rightarrow S^1$  be Morse functions on manifolds  $N_1, N_2$ . Let  $F : N_1 \times N_2 \rightarrow N_2$  be a map, such that for each  $a \in N_2$  the map  $x \mapsto F(a, x)$  is a diffeomorphism  $N_1 \rightarrow N_1$ . Define a function  $g : N_1 \times N_2 \rightarrow S^1$  by the following formula:*

$$g(x_1, x_2) = g_1(x_1) + g_2(F(x_1, x_2)).$$

*Then  $g$  is a Morse function,  $\text{Crit}(g) = \text{Crit}(g_1) \times \text{Crit}(g_2)$  and for every  $a_1 \in \text{Crit}(g_1)$ ,  $a_2 \in \text{Crit}(g_2)$  we have  $\text{ind}(a_1, a_2) = \text{ind}(a_1) + \text{ind}(a_2)$ .*

*Proof.* Define a function  $g_0$  on  $N_1 \times N_2$  by the following formula

$$g_0(x_1, x_2) = g_1(x_1) + g_2(x_2).$$

The conclusions of our Lemma hold obviously if we replace  $g$  by  $g_0$  in the statement of the Lemma. Observe now that the function  $g$  is diffeomorphic to  $g_0$  via the diffeomorphism

$$(x_1, x_2) \mapsto (x_1, F(x_1, x_2)).$$

The lemma follows.  $\square$

**Corollary 2.3.** *Let  $K \subset S^3$  be a classical knot, denote by  $S(K)$  the spun knot of  $K$ . Then*

$$(5) \quad \mathcal{MN}(S(K)) \leq 2\mathcal{MN}(K)$$

*Proof.* In this case  $M = S^1$  and  $[\lambda] = 0$ . We have  $\mathcal{MN}(S^1, 0) = 2$  and the result follows.  $\square$

The classical theorems concerning fibrations of spun knots follow from Theorem 2.1:

**Corollary 2.4.** (D. Roseman [12]) *If  $K$  is fibred, then  $\mathcal{MN}(\sigma(M, K))$  is fibred.*

*Proof.* Since  $\mathcal{MN}(K) = 0$ , Theorem 2.1 implies  $\mathcal{MN}(\sigma(M, K)) = 0$ .  $\square$

**Corollary 2.5.** (E.C. Zeeman [14]) *The  $d$ -twist spun knot of any classical knot  $K$  is fibred for  $d \geq 1$ .*

*Proof.* Consider a great circle  $\Sigma$  in  $S^2$ . The  $d$ -twist spun knot of  $K$  is by definition the  $n+1$ -knot  $\sigma(\Sigma, K, \lambda)$  in  $S^3$  where  $\Sigma \rightarrow \Sigma$  is a map of degree  $d$ . The assertion follows, since  $\mathcal{MN}(S^1, \lambda) = 0$ .  $\square$

**Remark 2.6.** The Zeeman's theorem above generalizes immediately to the following statement: If  $\mathcal{MN}(M, \lambda) = 0$ , then the knot  $\sigma(M, K, \lambda)$  is fibred for any knot  $K$ .

### 3. ROTATION

Let  $\Sigma$  be an equatorial sphere of  $S^{n+1}$ . We can view the sphere  $S^{n+1}$  as the union of two discs  $D_+ \cup D_-$  intersecting by  $\Sigma$ . Consider  $S^{n+1}$  as the equatorial sphere of  $S^{n+2}$ . The sphere  $S^{n+2}$  can be considered as the result of rotation of the disc  $D_+$  around its boundary  $\Sigma$ . We have the (linear orthogonal) action of  $S^1$  on  $S^{n+2}$ , such that  $\Sigma$  is the fixed point set of the action, and the action is free on the rest of the sphere  $S^{n+2}$ . Let  $K^{n-1}$  be an  $(n-1)$ -knot in  $S^{n+1}$ . We can assume

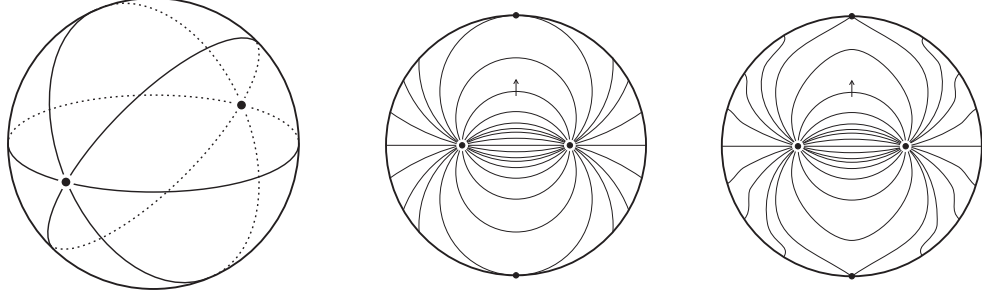


FIGURE 1.

that  $K^{n-1} \subset \text{Int } D_+$ . Rotation of  $K^{n-1}$  around  $\Sigma$  gives a submanifold  $R(K)$  of codimension 2 in  $S^{n+2}$ . The manifold  $R(K)$  is diffeomorphic to  $S^1 \times K$ . We call this construction *rotation*. When  $\dim K = 1$ , the manifold  $R(K)$  is sometimes called the *spun torus* of  $K$ .

In this section we relate the Morse-Novikov numbers of  $R(K)$  with those of  $K$ . The main aim of this section is to prove the following theorem.

**Theorem 3.1.**

$$\mathcal{MN}(R(K)) \leq 2\mathcal{MN}(K) + 2.$$

To prove the theorem we associate to each given regular Morse function  $\phi : S^{n+1} \setminus K^{n-1} \rightarrow S^1$  a regular Morse function  $R(\phi) : S^{n+2} \setminus R(K^{n-1}) \rightarrow S^1$  such that  $m(R(\phi)) = 2m(\phi) + 2$ .

We begin by an outline of this construction for the simplest case when  $n = 1$  and  $K$  consists of two points in  $S^2$  (Subsection 3.1). In Subsection 3.2 we give a detailed proof of the assertion of the theorem in full generality.

**3.1. Rotation of  $S^0$ .** Let  $K^0 = \{a, b\} \subset S^2$ . The manifold  $S^2 \setminus \{a, b\}$  is fibered over  $S^1$ , and the structure of the level lines of this fibration is shown on the figure 1 (left).

Let  $D_-$  be a small 2-disc around any regular point  $a$  of  $f$ . Denote by  $D_+$  the complement  $S^2 \setminus \text{Int } D_-$ , so that  $S^2 = D_+ \cup D_-$  and the discs  $D_\pm$  intersect by their common boundary  $\Sigma$ . Removing  $D_-$  we obtain a map  $f : D_+ \setminus \{a, b\} \rightarrow S^1$ ; the structure of its level lines is shown on the figure 1 (middle).

The restriction  $f|_\Sigma$  has two non-degenerate critical points:  $N$  and  $S$ . The vector  $v$  in the figure depicts the gradient of the map  $f$ . Applying the rotation construction to  $K_0$  we obtain a trivial 2-component link  $R(K^0)$  in  $S^3$ . Let  $F_0 : S^3 \setminus R(K_0) \rightarrow S^1$  be the unique  $S^1$ -invariant function such that  $F_0|_{D_+} = f$ . This function is continuous, but not smooth, since its level surfaces have conical singularities in the points of  $\Sigma$ . To repair this, we will modify the function  $f$  in a neighbourhood of  $\Sigma$  so that the level lines of the modified function  $g : D_+ \setminus \{a, b\} \rightarrow S^1$  are as depicted on the figure 1 (right).

Each non-singular level line intersecting  $\Sigma$  is orthogonal to  $\Sigma$  at the intersection point. Let  $G_0 : S^3 \setminus R(K_0) \rightarrow S^1$  be the unique  $S^1$ -invariant function such that  $G_0|_{D_+} = g$ . Then  $G_0$  is a  $C^\infty$  function having two critical points  $N$  and  $S$ . Observe that the descending disc of the critical point  $S$  of the function  $G_0|_\Sigma$  is

in  $\Sigma$ , therefore the descending discs of  $G_0$  will have the same dimension 1, and  $\text{ind}_{G_0} S = 1$ . The same reasoning holds for the *ascending disc* of the critical points  $N$ , therefore  $\text{ind}_{G_0} N = 2$ .

**3.2. The general case.** Let  $\Sigma$  be the unit sphere in  $\mathbb{R}^{n+2}$ , that is,

$$\Sigma = \{(x_0, \dots, x_{n+1}) \mid x_0^2 + \dots + x_{n+1}^2 = 1\}.$$

Denote by  $\Sigma'$  its intersection with the hyperplane  $x_{n+1} = 0$ . Let  $a = (0, \dots, 0, 1)$ ; for each point  $z \in \Sigma'$  denote by  $C(z)$  the great circle through  $a, -a, z$ , and by  $C'(z)$  the closed semicircle containing these three points. The projection  $p$  onto the  $(n+1)$ -th coordinate gives the bijection of  $C'(z)$  onto the closed interval  $[-1, 1]$ ; this bijection is a diffeomorphism when restricted to  $C'(z) \setminus \{a, -a\}$ . Let  $\beta : [-1, 1] \rightarrow [-1, 1]$  be a diffeomorphism such that  $\beta(x) = x$  for  $x$  in a neighbourhood of  $\pm 1$ . Then there is a unique diffeomorphism  $\tilde{\beta}$  of  $\Sigma$  onto itself such that for every  $z$  the curve  $C'(z)$  is  $\tilde{\beta}$ -invariant and  $p(\tilde{\beta}(v)) = \beta(p(v))$  for every  $v$ . The diffeomorphism  $\tilde{\beta}$  will be called *the sliding, associated to  $\beta$* . Observe that every sliding is isotopic to the identity map.

Let  $D_\rho \subset \Sigma$  be the geodesic disc of radius  $\rho$  centered in  $-a$ . Let

$$D_- = D_{\pi/2} = \{(x_0, \dots, x_{n+1}) \mid x_{n+1} \leq 0\},$$

$$D_+ = \{(x_0, \dots, x_{n+1}) \mid x_{n+1} \geq 0\}.$$

Put  $\Sigma_\rho = \partial D_\rho$ . Let  $N(\Sigma_\rho, \epsilon)$  denote the geodesic tubular neighbourhood of  $\Sigma_\rho$ . For a given  $\rho$  and  $\epsilon > 0$  sufficiently small there is a sliding  $\sigma$  sending  $D_\rho$  to  $D_-$  and sending each normal geodesic segment of length  $2\epsilon$  in  $N(\Sigma_\rho, \epsilon)$  isometrically to the corresponding normal geodesic segment in  $N(\Sigma, \epsilon)$ . We have therefore a commutative diagram

$$\begin{array}{ccc} N(\Sigma_\rho, \epsilon) & \xrightarrow{\sigma} & N(\Sigma, \epsilon) \\ \Phi \uparrow & & \uparrow \Psi \\ \Sigma_\rho \times ]-\epsilon, \epsilon[ & \xrightarrow{\bar{\sigma}} & \Sigma \times ]-\epsilon, \epsilon[ \end{array}$$

where the vertical arrows are diffeomorphisms and  $\bar{\sigma}(x, \tau) = (\sigma(x), \tau)$ .

Let  $K$  be an  $(n-1)$ -knot in  $S^{n+1}$  and  $\phi : S^{n+1} \setminus K \rightarrow S^1$  a Morse map. We can assume that

- 1)  $K \subset \text{Int } D_+$ .
- 2)  $-a \notin \text{Crit } \phi$ ,
- 3) the submanifold  $\phi^{-1}(\phi(-a))$  is tangent to the hyperplane defined by the equation  $x_n = 0$ .

The restriction  $\phi \mid \partial D_\rho$  can be considered as a real-valued Morse map. Choosing  $\rho$  sufficiently small we can assume that  $\phi \mid \Sigma_\rho$  is a Morse map having one maximum and one minimum. Denote the function  $\phi \circ \Phi$  by  $h : \Sigma_\rho \times ]-\epsilon, \epsilon[ \rightarrow \mathbb{R}$ . For  $\rho$  sufficiently small, this function has the following property:

$$(6) \quad \text{If } \frac{\partial h}{\partial t}(x, t) = 0, \text{ then } \frac{\partial h}{\partial x}(x, t) \neq 0, \text{ where } x \in \Sigma_\rho, t \in ]-\epsilon, \epsilon[.$$

Consider the restriction of  $\phi$  to the subset  $S^{n+1} \setminus (K \cup D_\rho)$ . Composing  $\phi$  with  $\sigma^{-1}$  we obtain a function

$$\phi_0 : D_+ \setminus \sigma(K) \rightarrow S^1.$$

This is a Morse map which extends to a geodesic tubular neighbourhood of  $\Sigma = \partial D^+$ , and can be considered as a real-valued Morse function in this neighbourhood. The restriction  $\phi_0|_{\Sigma}$  has two critical points of indices  $n$  and  $0$ . Denote these critical points by  $N$  and  $S$ , so that  $\text{ind}_{\phi_0} N = n$ ,  $\text{ind}_{\phi_0} S = 0$ . The function  $h_0 = \phi_0 \circ \Psi$  has the following property:

$$(7) \quad \text{If } \frac{\partial h_0}{\partial t}(x, t) = 0, \text{ then } \frac{\partial h_0}{\partial x}(x, t) \neq 0, \text{ where } x \in \Sigma, t \in ]-\epsilon, \epsilon[.$$

Now we will modify the function  $\phi_0$  nearby  $\Sigma$ . Let  $\lambda : [-\epsilon, \epsilon] \rightarrow \mathbb{R}$  be a  $C^\infty$  function such that  $\lambda(t) = |t|$  for  $t$  in a neighbourhood of  $\{-\epsilon, \epsilon\}$  and  $\lambda(t) = t^2$  for  $|t| \leq \epsilon/2$ . Define a function  $h_1$  by the following formula:

$$h_1(x, t) = h_0(x, \lambda(t)),$$

and define a function

$$\phi_1 : D_+ \setminus \sigma(K) \rightarrow S^1$$

as follows:

- 1) if  $v \notin N(\Sigma, \epsilon)$ , put  $\phi_1(v) = \phi_0(v)$ .
- 2) if  $v \in N(\Sigma, \epsilon)$ ,  $v = \Psi(x, t)$  with  $x \in \Sigma$ ,  $t \in ]-\epsilon, \epsilon[$ , put  $\phi_1(v) = h_1(x, t)$ .

**Proposition 3.2.** *The function  $\phi_1$  has two critical points in  $N(\Sigma, \epsilon)$ , namely  $N$  and  $S$ . Their indices are equal, respectively, to  $n$  and  $1$ .*

*Proof.* The partial derivatives of  $h_1$  are equal to  $\frac{\partial h_0}{\partial x}(x, \lambda(t))(x, t)$  and  $\frac{\partial h_0}{\partial t}(x, \lambda(t)) \cdot \lambda'(t)$ . For  $t = 0$  the second derivative equals  $0$ , and  $\frac{\partial h_1}{\partial x}(x, \lambda(t))(x, 0)$  vanishes in  $N$  and  $S$ . If  $t \neq 0$ , then  $\lambda'(t) \neq 0$ , and for  $(x, t)$  to be a critical point of  $\phi_1$  it is necessary that  $\frac{\partial h_0}{\partial t}(x, \lambda(t))(x, t)$  vanish, which implies that  $\frac{\partial h_0}{\partial t}(x, t)(x, t) \neq 0$  (see the property (7)).  $\square$

Now we are ready to construct a Morse function on the complement to  $R(K)$ . Observe that the knot  $K$  is equivalent to the knot  $\sigma(K)$ . By a certain abuse of notation we will replace  $\sigma(K)$  by  $K$ , so in particular,  $K \subset \text{Int } D_+$ . Add one more coordinate  $x_{n+2}$  and consider the sphere

$$\Sigma = \{(x_0, \dots, x_{n+2}) \mid x_0^2 + \dots + x_{n+2}^2 = 1\}.$$

We have  $D_+ \subset S^{n+2}$ . The knot  $R(K)$  is defined by the following formula:

$$R(K) = \{(x_0, \dots, x_{n+2}) \mid (x_0, \dots, x_n, \sqrt{x_{n+1}^2 + x_{n+2}^2}) \in K\}.$$

The circle  $S^1$  acts on  $S^{n+2}$  by rotations in the two last coordinates. Define the Morse function  $\phi_2$  on the complement to  $R(K)$  by the two following properties:

- 1)  $\phi_2|_{D_+ \setminus K} = \phi_1$ .
- 2)  $\phi_2$  is  $S^1$ -invariant.

The second property implies that

$$\phi_2(x_0, \dots, x_{n+2}) = \phi_1\left(x_0, \dots, x_n, \sqrt{x_{n+1}^2 + x_{n+2}^2}\right).$$

Observe that the property 2) of the function  $\phi_1$  guarantees that  $\phi_2$  is  $C^\infty$  on the subset  $S^{n+2} \setminus R(K)$ .

**Proposition 3.3.** 1)  $\text{Crit}(\phi_2) = S^1 \cdot \text{Crit}(\phi_1) \cup \{N, S\}$ ,



2) *The critical points  $N$  and  $S$  are non-degenerate, and*

$$\text{ind}_{\phi_2} N = \text{ind}_{\phi_1} N = n, \quad \text{ind}_{\phi_2} S = \text{ind}_{\phi_1} S + 1 = 2.$$

*Proof.* The point 1) is easy to deduce from the definition of  $\phi_2$ . As for the indices of the critical points observe that the descending disc of the critical point  $N$  in  $N(\Sigma, \epsilon)$  belongs to the sphere  $\Sigma$  which is fixed by the action of  $S^1$ . Thus the index of  $N$  does not change when we replace  $\phi_1$  by  $\phi_2$ . A similar argument applies to the *ascending disc* of  $S$ , and this implies the rest of the proposition.  $\square$

Each critical point of  $\phi_1$  gives rise to a circle of critical points of  $\phi_2$ . Using the same method, as in the previous work of the authors, we wwe perturb the function  $\phi_2$  in a neighbourhood of each of these critical circles, and obtain finally a regular Morse function  $R(\phi)$  on the complement to  $R(K)$  such that

$$\# \text{Crit}(R(\phi)) = 2\# \text{Crit}(\phi_2) + 2.$$

This completes the proof of Theorem 3.1.  $\square$

**3.3. 4-thread spinning.** In this subsection we give a brief description of one more construction of surface-links. Let  $L \subset S^3$  be a classical link and  $\phi : S^3 \setminus L \rightarrow S^1$  a Morse map. Let  $p, q \in L$  and let  $\gamma : [0, 1] \rightarrow S^3$  be a  $C^\infty$  curve joining  $p$  and  $q$  and belonging entirely to one of the regular level surfaces  $\phi^{-1}(\lambda)$  of the map  $\phi$ . We assume moreover that  $\text{Im } \gamma \cap L = \{p, q\}$ , and that  $\gamma'(0)$  and  $\gamma'(1)$  are not tangent to  $L$ . Let  $D$  be a small neighbourhood of  $\text{Im } \gamma$  diffeomorphic to a 3-disc. Denote by  $\Sigma$  its boundary. We can assume that  $L \cap \Sigma$  consists of four points and that  $L$  is orthogonal to  $\Sigma$  at each of these points. Denote by  $S_0^2$  the 2-sphere with 4 points removed. Recall that there is a standard Morse function  $\phi_0$  on  $S_0^2$  having 2 critical points of indices 1. We can assume that the restriction of  $\phi$  to  $\Sigma \setminus L$  is diffeomorphic to  $\phi_0$ .

Remove the interior of  $D$  from  $S^3$  and rotate the remaining manifold  $S^3 \setminus \text{Int } D$  around  $\Sigma$ . We obtain the sphere  $S^4$ ; the subset which is spun by  $L \setminus \text{Int } D$  during the rotation is an embedded 2-surface in  $S^4$ .

We call this construction *4-thread spinning* to distinguish it from the usual spinning, and denote the resulting surface-link by  $S'(L)$ . If  $p$  and  $q$  are on different connected components of  $L$ , then the number of connected components of  $S'(L)$  is the same as for  $L$ . If  $p$  and  $q$  are in different connected components of  $L$ , then the number of connected components of  $S'(L)$  equals that of  $L$  increased by 1. Applying the same method as in the Subsection 3.2 we can construct a Morse function  $\tilde{\phi}$  on  $S^4 \setminus S'(L) \rightarrow S^1$  such that  $m(\tilde{\phi}) = m(\phi) + 2$ .

**Corollary 3.4.**

$$\mathcal{MN}(S'(L)) \leq 2\mathcal{MN}(L) + 2.$$

$\square$

#### 4. SURFACE-LINKS

In this section we develop circle-valued Morse theory for surface-links.

**4.1. Motion pictures and saddle numbers.** Let  $F$  be a surface-link, that is, a closed oriented 2-dimensional  $C^\infty$  submanifold of  $S^4$ . We can assume  $F \subset \mathbb{R}^4$ .

Choose a projection  $p$  of  $\mathbb{R}^4$  onto a line. Assume that the critical points of the function  $p|_F$  are non-degenerate. Denote by  $sdl(F)$  the minimal number of saddle points of  $p|_F$  over all the projections  $p$ .

**Definition 4.1.** A saddle number  $sd(F)$  is the minimum of numbers  $sdl(F')$  where  $F'$  ranges over all surface-links  $F'$  ambiently isotopic to  $F$ .

The invariant  $sd(F)$  is closely related to the *ch-index* of  $F$ , introduced and studied by K. Yoshikawa in [13]. In particular, we have  $sd(F) \leq ch(F)$ . In order to relate the number  $sd(F)$  to  $\mathcal{MN}(K)$  we will reformulate the definition of the saddle number.

Let  $F \subset S^4$  be a surface-link. The equatorial 3-sphere  $\Sigma^3$  of the standard Euclidean sphere  $S^4$  divides  $S^4$  into two parts:

$$S^4 = D_+^4 \cup D_-^4, \text{ with } D_+^4 \cap D_-^4 = \Sigma^3.$$

We assume that  $F$  is included in  $\text{Int}(D_+^4)$  and  $F$  does not include the centre of  $D_-^4$ . Perturbing the embedding  $F \subset D_-^4$  if necessary, we can assume that the restriction  $\rho = r|_F$  of the radius function  $r : D_-^4 \rightarrow [0, 1]$  is a Morse function. The family  $\{(r^{-1}(t), \rho^{-1}(t))\}_{t \in [0, 1]}$  of possibly singular links can be drawn as a *motion picture* (see [8], Chapter 8). Each singularity of a link in the family corresponds to a critical point of  $\rho$ . A critical point of  $\rho$  of index 0 (1, 2, respectively) is called *minimal point* (*saddle point*, *maximal point*, respectively) of  $\rho$ , which is represented by a *minimal band* (*saddle band*, *maximal band*, respectively) in (a modification of) the motion picture.

It is clear that the minimal number of the saddle points for all such Morse functions  $\rho$  is equal to  $sd(F)$ .

**Theorem 4.2.**  $\mathcal{MN}(F) \leq 2sd(F) + \chi(F) - 2$ .

*Proof.* Since  $\rho$  is a Morse function, the manifold  $D_-^4 \setminus \text{Int } N(F)$  admits a handle decomposition with one 0-handle and  $m_i(\rho)$   $(i + 1)$ -handles for  $i \in \{0, 1, 2\}$  (see [7], and also [6], Proposition 6.2.1).

The exterior  $E(F) = S^4 \setminus \text{Int } N(F)$  of  $F$  is obtained by attaching a 4-handle  $D_+^4$  to  $D_-^4 \setminus \text{Int } N(F)$ . Since  $D_-^4 \setminus \text{Int } N(F)$  is connected, there is a 3-handle in  $D_-^4 \setminus \text{Int } N(F)$  which connects  $\partial N(F)$  with  $\partial D_-^4$ . Thus the 3-handle cancels the 4-handle  $D_+^4$  (see [9], Section 5). Turning the handlebody upside down, we obtain a dual decomposition of  $E(F)$  and a corresponding Morse function  $f : E(F) \rightarrow \mathbb{R}$  which is constant on  $\partial E(F)$  and the following Morse numbers:  $m_1(f) = m_2(\rho) - 1$ ,  $m_2(f) = m_1(\rho)$ ,  $m_3(f) = m_0(\rho)$ ,  $m_4(f) = 1$ .

Using the argument from work of the second author [10], p. 629, we can deform the real-valued Morse function  $f$  to a circle-valued regular function  $\phi : E(F) \rightarrow S^1$ , such that  $m_k(f) = m_k(\phi)$  for every  $k$ . Consider the function  $-\phi$ , which has one critical point of index 0. Applying the cancellation of this local minimum, we obtain a Morse function  $\psi : E(F) \rightarrow S^1$  belonging to the class  $-\xi$ , and such that  $m_0(\psi) = 0$ ,  $m_1(\psi) = m_3(f) - 1$ ,  $m_2(\psi) = m_2(f)$ ,  $m_3(\psi) = m_1(f)$ ,  $m_4(\psi) = 0$ . Put  $g = -\psi$ . Then we have

$$m_0(g) = m_4(g) = 0, \quad m_1(g) = m_2(\rho) - 1,$$

$$m_2(g) = m_1(\rho), \quad m_3(g) = m_0(\rho) - 1.$$

Observe that  $m_0(\rho) - m_1(\rho) + m_2(\rho) = \chi(S^2) = 2$ , therefore the total number of critical points of  $g$  equals  $2m_1(\rho)$ . Choosing the function  $\rho$  with  $m_1(\rho) = sd(F)$  we accomplish the proof.  $\square$

**Corollary 4.3.** Let  $K \subset S^4$  be a 2-knot. Then  $\mathcal{MN}(C_K) \leq 2sd(K)$ .  $\square$

**Proposition 4.4.** *Let  $F \subset S^4$  be the trivial  $k$ -component surface-link. Then  $\mathcal{MN}(F) = 4k - 2 - \chi(F)$ .*

*Proof.* It is not difficult to show that  $\hat{b}_1(C(F)) \geq k - 1$ ,  $\hat{b}_3(C(F)) \geq k - 1$ . Therefore for every regular Morse map  $f : C(F) \rightarrow S^1$  we have  $m_1(f) + m_3(f) \geq 2(k - 1)$ . Assuming  $m_0(f) = m_4(f) = 0$  we have  $m_1(f) - m_2(f) + m_3(f) = 2 - \chi(F)$ , and  $\mathcal{MN}(C(F)) \geq 4k - 2 - \chi(F)$ ; this lower bound coincides with the upper bound derived from Theorem 4.2.  $\square$

**4.2. Spun knots.** Let  $K$  be a classical knot in  $S^3$  denote by  $S(K)$  the corresponding spun knot.

**Proposition 4.5.** *If  $K$  is a non-fibered knot of tunnel number 1, then  $\mathcal{MN}(S^4 \setminus S(K)) = 4$ .*

*Proof.* Recall that  $\mathcal{MN}(S^4 \setminus S(K)) \leq 2\mathcal{MN}(K)$  (Corollary 2.3). In the paper [10] of the second author it is shown that  $\mathcal{MN}(K) \leq 2t(K)$ , hence  $\mathcal{MN}(S(K)) \leq 4$  by Corollary 2.3. Put  $G = \pi_1(S^3 \setminus K)$ , then  $\pi_1(S^4 \setminus S(K))$ ; let  $H = [G, G]$ . Let  $f : S^4 \setminus S(K) \rightarrow S^1$  be a regular Morse map without minima and maxima. If  $m_1(f) = 0$ , then a standard Morse-theoretic argument applied to the infinite cyclic cover of  $S^4 \setminus S(K)$  implies that  $H$  is finitely generated, which is impossible, since  $K$  is not fibered. Therefore  $m_1(f) \geq 1$ , and similarly,  $m_3(f) \geq 1$ , hence  $m_2(f) \geq 2$  and the proposition is proved.  $\square$

**4.3. Surface-links of Yoshikawa's table.** Yoshikawa [13] suggested a method for enumerating surface-links. To each surface-link  $F$  he associated a natural number  $ch(F)$ . His methods allowed him to make a list of all (weakly prime) surface-links  $F$  with  $ch(F) \leq 10$ . It is clear from the definition of the invariant  $ch(F)$  that we have  $sd(F) \leq ch(F)$ . In the rest of this section we assume that the reader is familiar with Yoshikawa's work, and with his terminology. There are 6 two-knots in Yoshikawa's table, namely

$$0_1, 8_1, 9_1, 10_1, 10_2, 10_3.$$

The trivial 2-knot  $0_1$  is obviously fibered. The knots  $8_1$  and  $10_1$  are spun knots of the trefoil knot and respectively of the figure 8 knot, thus both  $8_1$  and  $10_1$  are fibered by [1].

The case of  $9_1$  is more complicated. The saddle number of this 2-knot is 2. Therefore  $\mathcal{MN}(9_1) \leq 4$ . Using the presentation of the fundamental group of the complement to  $9_1$  (see [13]) and Poincaré duality properties it is easy to compute the Novikov numbers of  $9_1$ . Namely we have  $\hat{q}_1 = 1, \hat{q}_2 = \hat{q}_3 = 0$ . Therefore

$$2 \leq \mathcal{MN}(9_1) \leq 4.$$

The 2-knot  $10_2$  is the 2-twist-spun knot of the trefoil knot, hence fibered by Zeeman's theorem [14]. Similarly,  $10_3$  is fibered, being the 3-twist spun of the trefoil knot.

The surface-link  $6_1^{0,1}$  is the result of spinning of the Hopf link which is fibered (see the left of Figure 2) therefore  $\mathcal{MN}(6_1^{0,1}) = 0$ .

The surface-link  $8_1^{1,1}$  is the spun torus of the Hopf link. Applying Theorem 3.1 we get the upper bound  $\mathcal{MN}(8_1^{1,1}) \leq 2$ . Computing the Euler characteristic implies the inverse inequality, so  $\mathcal{MN}(8_1^{1,1}) = 2$ .

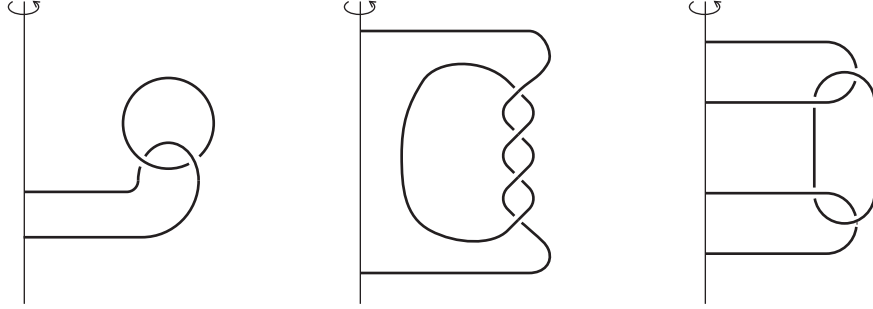


FIGURE 2.

The same argument applies to the surface-link  $10_1^1$ , which is the spun torus of the trefoil knot, see the figure 2 (middle), so that  $\mathcal{MN}(10_1^1) = 2$ .

The surface-link  $10_1^{0,1}$  is the result of spinning of the link  $4_1^2$  which is fibred, therefore  $\mathcal{MN}(10_1^{0,1}) = 0$ .

The case of the surface-link  $F = 10_1^{0,0,1}$  is more complicated. This surface-link is the result of 4-threaded spinning of the connected sum  $L$  of two copies of the Hopf link, see Figure 2 (right) and applying Corollary 3.4 we deduce  $\mathcal{MN}(F) \leq 2$ . The computation of Euler characteristic gives the lower bound 2 for the Morse-Novikov number, thus  $\mathcal{MN}(10_1^{0,0,1}) = 2$ .

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